

The Space-like Surfaces with Vanishing Conformal Form in the Conformal Space ^{*†}

Changxiong Nie

Abstract. The conformal geometry of surfaces in the conformal space \mathbf{Q}_1^n is studied. We classify the space-like surfaces in \mathbf{Q}_1^n with vanishing conformal form up to conformal equivalence.

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1 Introduction.

In [1] Wang gave the structure equations for Möbius geometry of submanifolds in the unit sphere. Three fundamental tensors \mathbb{A} , \mathbb{B} and Φ arise naturally in the structure equations. In [1] \mathbb{A} is called the Blaschke tensor, \mathbb{B} the Möbius second fundamental form, and Φ the Möbius form. Together with Möbius metric g , these tensors determine the submanifold up to Möbius transformations of the unit sphere.

Li and Wang^[2] classified surfaces with vanishing Möbius form in sphere space \mathbb{S}^{n+1} . Readers should be reminded that Bryant^[3] have classified all minimal surfaces with constant curvature in the unit sphere \mathbf{S}^n , the hyperbolic space \mathbf{H}^n and Euclidean space \mathbf{R}^n . Li and Wang used Bryant's results. It is interesting to classify stationary surfaces with constant curvature in \mathbf{R}_1^n , \mathbf{S}_1^n or \mathbf{H}_1^n . Some other results about Lorentz conformal geometry see refs. [4-7]. Further relative knowledge refers to [8-10].

We conclude this paper with

The Main Theorem Let $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ be a regular spacelike full surface with vanishing form. Then x are the following four alternatives:

- (i) x is a stationary surface with constant curvature in \mathbf{R}_1^n , \mathbf{S}_1^n or \mathbf{H}_1^n .
- (ii) x is a hyperbolic cylinder $\mathbf{H}^1 \times \mathbf{R}$ in \mathbf{R}_1^3 .
- (iii) x is a surface $\mathbf{H}^1(\sqrt{r}) \times \mathbf{S}^1(\sqrt{1+r})$ in \mathbf{S}_1^3 , $r > 0$.
- (iv) x is a hyperbolic torus $\mathbf{H}^1(\sqrt{1+r}) \times \mathbf{H}^1(\sqrt{-r})$ in \mathbf{H}_1^3 , $-\frac{1}{2} \leq r < 0$.

2. The fundamental equations

^{*}Faculty of Mathematics and Computer Sciences, Hubei University, Wuhan 430062, People's Republic of China (e-mail: chxn timer@163.com).

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Let \mathbb{R}_s^n be the real vector space \mathbb{R}^n with the Lorentzian inner product \langle, \rangle given by

$$\langle X, Y \rangle := \sum_{i=1}^{n-s} x_i y_i - \sum_{i=n-s+1}^n x_i y_i,$$

where $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We define the de Sitter sphere \mathbf{S}_1^n and anti-de Sitter sphere \mathbf{H}_1^n by

$$\mathbf{S}_1^n = \{u \in \mathbf{R}_1^{n+1} | \langle u, u \rangle = 1\}, \quad \mathbf{H}_1^n = \{u \in \mathbf{R}_2^{n+1} | \langle u, u \rangle = -1\}.$$

We call Lorentzian space \mathbf{R}_1^n , de Sitter sphere \mathbf{S}_1^n and anti-de Sitter sphere \mathbf{H}_1^n Lorentzian space forms. Denote

$$\mathbf{Q}_1^n = \{[x] \text{ is the projective coordinates} | \langle x, x \rangle = 0, x \in \mathbf{R}_2^{n+2}\}.$$

By some conformal diffeomorphisms we may regard \mathbf{Q}_1^n as the common compactified space of $\mathbf{R}_1^n, \mathbf{S}_1^n$ and \mathbf{H}_1^n . In fact the conformal space \mathbf{Q}_1^n has a standard Lorentzian metric. We research the conformal geometry of surfaces under the conformal group of this Lorentzian metric. We refer the reader to [4, 7] for further details.

Suppose that $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ is a space-like surface. That is, $x_*(\mathbf{TM})$ is non-degenerated subbundle of \mathbf{TQ}_1^n . Let $y : U \rightarrow \mathbf{R}_2^{n+2}$ be a lift of $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ defined in an open subset U of \mathbf{M} . We denote by Δ and κ Laplacian and the normalized scalar curvature of the local positive definite metric $\langle dy, dy \rangle$. Then we have

Theorem 2.1. (see [1] Theorem 1.2.) On \mathbf{M} the 2-form $g = -(\langle \Delta y, \Delta y \rangle - 4\kappa) \langle dy, dy \rangle$ is a globally defined invariant of $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ under the Lorentz group transformations of \mathbf{Q}_1^n .

Let $x : \mathbf{M}^2 \rightarrow \mathbf{Q}_1^n$ be a *regular* space-like surface. That is, the 2-form $g = -(\langle \Delta y, \Delta y \rangle - 4\kappa) \langle dy, dy \rangle$, which is called conformal metric, is non-degenerated. Let $Y = \sqrt{-(\langle \Delta y, \Delta y \rangle - 4\kappa)} y$ be the canonical lift of x and define $N : \mathbf{M} \rightarrow \mathbf{R}_2^{n+2}$ by $N = -\frac{1}{2} \Delta Y - \frac{1}{8} \langle \Delta Y, \Delta Y \rangle Y$. Let $\{E_\alpha\}$ be a local basis of the conformal normal bundle \mathbf{V} of x . If $z = u + iv$ be a local isothermal coordinate on \mathbf{M} for g , we can write $g = e^{2\omega} |dz|^2 = \frac{1}{2} e^{2\omega} (dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ for some local smooth function ω . Denote by K the Gauss curvature of g , we have $\Delta Y = 4e^{-2\omega} Y_{z\bar{z}}, K = -4e^{-2\omega} \omega_{z\bar{z}}$. Since $\{Y, N, \text{Re}(Y_z), \text{Im}(Y_z), E_\alpha\}$ is a moving frame in \mathbf{R}_2^{n+2} along \mathbf{M} , one can write the structure equations and the fundamental equations as in [2]. Define

$$\psi = 2\langle N_z, Y_z \rangle, \quad \phi_\alpha = \langle N_z, E_\alpha \rangle, \quad \Omega_\alpha = 2\langle Y_{zz}, E_\alpha \rangle, \quad A_{\alpha\beta} = \langle (E_\alpha)_z, E_\beta \rangle, \quad (2.1)$$

and $\phi^\alpha = \sum_\beta g^{\alpha\beta} \phi_\beta$, $\Omega^\alpha = \sum_\beta g^{\alpha\beta} \Omega_\beta$, $A_\alpha^\beta = \sum_\gamma g^{\beta\gamma} A_{\alpha\gamma}$. The structure equations are

$$N_z = \frac{1}{8} (4K - 1) Y_z + e^{-2\omega} \psi Y_{\bar{z}} + \sum_\alpha \phi^\alpha E_\alpha, \quad (2.2)$$

$$Y_{zz} = -\frac{1}{2} \psi Y + 2\omega_z Y_z + \frac{1}{2} \sum_\alpha \Omega^\alpha E_\alpha, \quad Y_{z\bar{z}} = -\frac{1}{16} e^{2\omega} (4K - 1) Y - \frac{1}{2} e^{2\omega} N, \quad (2.3)$$

$$(E_\alpha)_z = -\phi_\alpha Y - e^{-2\omega} \Omega_\alpha Y_{\bar{z}} + \sum_{\beta} A_\alpha^\beta E_\beta. \quad (2.4)$$

The fundamental equations are

$$\psi_{\bar{z}} = \frac{1}{2} e^{2\omega} K_z - \sum_{\alpha} \Omega^\alpha \bar{\phi}_\alpha, \quad \sum_{\alpha} \Omega^\alpha \Omega_\alpha = -\frac{1}{4} e^{4\omega}, \quad (\Omega_\alpha)_{\bar{z}} = -\sum_{\beta} \Omega^\beta \bar{A}_{\beta\alpha} - e^{2\omega} \phi_\alpha, \quad (2.5)$$

$$(\phi_\alpha)_{\bar{z}} - \frac{1}{2} e^{-2\omega} \bar{\psi} \Omega_\alpha + \sum_{\beta} \phi^\beta \bar{A}_{\beta\alpha} = (\bar{\phi}_\alpha)_z - \frac{1}{2} e^{-2\omega} \psi \bar{\Omega}_\alpha + \sum_{\beta} \bar{\phi}^\beta A_{\beta\alpha}, \quad (2.6)$$

$$(A_{\alpha\beta})_{\bar{z}} - (\bar{A}_{\alpha\beta})_z = \frac{1}{2} e^{-2\omega} (\Omega_\alpha \bar{\Omega}_\beta - \bar{\Omega}_\alpha \Omega_\beta) + \sum_{\gamma} (\bar{A}_{\alpha\gamma} A_\beta^\gamma - A_{\alpha\gamma} \bar{A}_\beta^\gamma). \quad (2.7)$$

Remark 2.1. $\Psi = \psi dz \otimes dz$, $\Phi = \sum_{\alpha} (\phi^\alpha dz + \bar{\phi}^\alpha d\bar{z}) \otimes E_\alpha$ and $\Omega = \sum_{\alpha} \Omega^\alpha dz \otimes dz \otimes E_\alpha$ are globally defined conformal invariants.

Remark 2.2. The Willmore equations are

$$(\phi_\alpha)_{\bar{z}} - \frac{1}{2} e^{-2\omega} \bar{\psi} \Omega_\alpha + \sum_{\beta} \phi^\beta \bar{A}_{\beta\alpha} = 0, \forall \alpha. \quad (2.8)$$

3. The classification of space-like surfaces in \mathbf{Q}_1^n with $\Phi = 0$

Let $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ be a space-like surface in \mathbf{Q}_1^n with vanishing conformal form, i.e., $\phi_\alpha = 0, 3 \leq \alpha \leq n$. Then the fundamental equations come into

$$\psi_{\bar{z}} = \frac{1}{2} e^{2\omega} K_z, \quad \bar{\psi} \Omega_\alpha = \psi \bar{\Omega}_\alpha, \quad e^{-4\omega} \sum_{\alpha} \Omega^\alpha \Omega_\alpha = -\frac{1}{4}, \quad (3.1)$$

$$(\Omega_\alpha)_{\bar{z}} = -\sum_{\beta} \Omega^\beta \bar{A}_{\beta\alpha}, \quad A_{\alpha\beta} = -A_{\beta\alpha}. \quad (3.2)$$

It follows from (3.2) that

$$\left(\sum_{\alpha} \Omega^\alpha \Omega_\alpha \right)_{\bar{z}} = 0, \quad (3.3)$$

thus the globally defined 4-form $\sum_{\alpha} \Omega^\alpha \Omega_\alpha dz^4$ is holomorphic on \mathbf{M} . From (3.1) we get

$$\bar{\psi} \sum_{\alpha} \Omega^\alpha \Omega_\alpha = \sum_{\alpha} \Omega^\alpha \bar{\Omega}_\alpha = -\frac{1}{4} e^{4\omega} \psi. \quad (3.4)$$

Immediately we have

Lemma 3.1. Let $x : \mathbf{M} \rightarrow \mathbf{Q}_1^n$ be a surface in \mathbf{Q}_1^n with vanishing conformal form. Then the conformal invariant $\Psi = \psi dz^2$ is holomorphic on \mathbf{M} . Then Ψ vanishes identically or the zero points of Ψ are isolated.

First we consider the case that $\Psi \equiv 0$. Thus from (3.1) K must be a constant. Then we get from (2.3) that

$$N = \frac{1}{8}(4K - 1)Y + \mathbf{c}$$

for some constant vector $\mathbf{c} \neq 0$ in \mathbf{R}_2^{n+2} . Therefore x is a conformal isotropic surface, i.e., x is stationary surface with constant curvature in $\mathbf{R}_1^n, \mathbf{S}_1^n$, or \mathbf{H}_1^n (see ref. [7]).

Now we come to discuss the case that the zero points of Ψ are isolated. In this case we can cut \mathbf{M} by some disjoint curves C_i to get a simply connected domain $\mathbf{U} = \mathbf{M} \setminus \sum_i C_i$ such that $x : \mathbf{U} \rightarrow \mathbf{Q}_1^n$ is a surface with $\Psi \neq 0$ on U . By choosing complex coordinate if necessary, we may assume that $\psi \equiv 1$ on \mathbf{U} . It follows from (2.1) and (3.1) that $K = -4e^{-2\omega}\omega_{z\bar{z}} = 0$. By (3.3) we know that $\{\Omega_\alpha\}$ are real functions. We define a global real vector field $E \in \mathbf{V}$ by

$$E = 2e^{-2\omega} \sum_a \Omega^a E_a, \quad (3.5)$$

then by (3.2) we have $\langle E, E \rangle = -1$. Choosing $\tilde{E}_3 = E$ and expanding it to a local orthonormal basis of the conformal normal bundle $\{\tilde{E}_3, \tilde{E}_4, \dots, \tilde{E}_n\}$, one can easily verify that

$$\tilde{\Omega}_3 = -\frac{1}{2}e^{2\omega}, \tilde{\Omega}_4 = \dots = \tilde{\Omega}_n = 0. \quad (3.6)$$

Using (3.3), we get

$$(\tilde{\Omega}_3)_{\bar{z}} = -\sum_{\beta=3}^n \tilde{\Omega}^\beta \bar{A}_{\beta 3} = -\tilde{\Omega}_3 \bar{A}_{33} = 0, \quad (3.7)$$

$$0 = (\tilde{\Omega}_\alpha)_{\bar{z}} = -\sum_{\beta \neq \alpha} \tilde{\Omega}^\beta \bar{A}_{\beta \alpha} = -\tilde{\Omega}_3 \bar{A}_{3\alpha}, \alpha > 3. \quad (3.8)$$

Thus ω is a constant and $A_{3\alpha} = 0, \forall \alpha$. Now the structure equations read

$$N_z = \frac{1}{8}Y_z + e^{-2\omega}Y_{\bar{z}}, \quad (3.9)$$

$$Y_{zz} = -\frac{1}{2}Y - \frac{1}{4}e^{2\omega}\tilde{E}_3, \quad Y_{z\bar{z}} = \frac{1}{16}e^{2\omega}Y - \frac{1}{2}e^{2\omega}N, \quad (3.10)$$

$$(\tilde{E}_3)_z = \frac{1}{2}Y_{\bar{z}}, \quad (\tilde{E}_\alpha)_z = \sum_{\beta} A_\alpha^\beta \tilde{E}_\beta, \quad \alpha > 3. \quad (3.11)$$

A surface is said to be full in \mathbf{Q}_1^n if $x(\mathbf{M})$ does not lie in any totally umbilic \mathbf{Q}_1^{n-1} of \mathbf{Q}_1^n . We assume that $n \geq 4$. Then fixing a point $p \in U$, we can find a constant vector $\xi \in \mathbf{R}_2^{n+2}$ with $\langle \xi, \xi \rangle = 1$ such that

$$\begin{aligned} \langle Y(p), \xi \rangle &= 0, & \langle N(p), \xi \rangle &= 0, & \langle Y_u(p), \xi \rangle &= 0, \\ \langle Y_v(p), \xi \rangle &= 0, & \langle \tilde{E}_3(p), \xi \rangle &= 0. \end{aligned} \quad (3.12)$$

We define real functions

$$\begin{aligned} f_1 &= \langle Y, \xi \rangle, \quad f_2 = \langle N, \xi \rangle, \quad f_3 = \langle Y_u, \xi \rangle, \\ f_4 &= \langle Y_v, \xi \rangle, f_5 = \langle \tilde{E}_3, \xi \rangle. \end{aligned} \quad (3.13)$$

Then by (3.11)-(3.13) we can find constants $\{a_{\lambda\mu}\}$ and $\{b_{\lambda\mu}\}$ such that

$$(f_\lambda)_u = \sum_{\mu} a_{\lambda\mu} f_\mu, \quad (f_\lambda)_v = \sum_{\mu} b_{\lambda\mu} f_\mu, \quad 1 \leq \lambda, \mu \leq 5. \quad (3.14)$$

By (3.12) and the uniqueness of the linear PDE (3.14) we get $f_\lambda \equiv 0$. In particular, $f_1 = \langle Y, \xi \rangle = 0$ on U , which implies that $\langle Y, \xi \rangle = 0$ on \mathbf{M} . If x is full then n must be 3.

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References

1. Wang C.P., Moebius geometry of submanifolds in S^n , Manuscripta Math. **96**(1998): 517-534
2. Li H.Z., Wang C.P., Surfaces with vanishing Moebius form in S^n , Acta Math. Sinica, Engl. Ser., **19**(2003): 671-678
3. Bryant R.L., Minimal surfaces of constant curvature in S^n , Trans. Amer. Math. Soc. **290**(1985): 259-271
4. Deng Y.J., Wang C.P., Willmore surfaces in Lorentzian space, Sci. China Ser. A, **35**(2005): 1361-1372
5. Nie C.X., Ma X., Wang C.P., Conformal CMC-surfaces in Lorentzian space forms, Chin. Ann. Math., Ser. B, **28** (2007): 299-310
6. Alias L.J., Palmer B., Conformal geometry of surfaces in Lorentzian space forms, Geometriae Dedicata, **60**(1996): 301-315
7. Nie C.X., Li T.Z., He Y.J., Wu C.X., Conformal isoparametric hypersurfaces with two distinct conformal principal curvatures in conformal space, Sci. China Ser. A, **53** (2010): 953-965
8. Blaschke W., Vorlesungen über Differentialgeometrie, Vol. 3, Springer, Berlin, 1929
9. Hertrich-Jeromin U., Introduction to Möbius Differential Geometry, London Math. Soc. Lecture Note Series, Vol. 300, Cambridge University Press, Cambridge, 2003
10. O'Neill B., Semi-Riemannian Geometry with Applications to Relativity, Pure and Applied Mathematics, 103, Academic Press, New York, 1983